# The Logic of Boolean Connectives

## Truth Tables, Tautologies & Logical Truths

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## Announcements

For 09.15.11

1. HW1 was due on Tuesday
   - If you didn't turn it in, it's late
2. HW2 & HW3 are due next Tuesday (09.20)
   - Electronic HW must be submitted before class
   - Written HW must be handed in at beginning of class
   - Otherwise, it's late
3. Optional sections have been scheduled
   - Wednesday 1:25-2:10, Uris 307
   - Thursday 1:25-2:10, Uris G22





## Outline







## Introduction

Truth Functions

- Last class, we learned the meaning of $\wedge, \vee, \neg$ in terms of truth functions
- We also saw that truth functions allowed us to do something useful:
  - Figure out the truth value of a complex sentence from the truth values of its atomic parts, and vice versa
- For example, we know that $\neg(\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b}))$ is true in a world where neither $a$ nor $b$ are cubes, since:
  - If $\neg(\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b}))$ is true, then $\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b})$ is false
  - If $\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b})$ is false, then $\mathsf{Cube}(\mathsf{a})$ and $\mathsf{Cube}(\mathsf{b})$ are false

# Introduction
## Truth Tables

- These calculations — ones like the one we just went through — are a bit clunky
- There's a more elegant method: Truth tables
- As it turns out, truth tables will also provide us with a helpful way to understand 3 core logical concepts:
  1. Logical Consequence
  2. Logical Truth
  3. Logical Equivalence
- Today, we'll learn all about truth tables & these applications!





# Review
## The Boolean Connectives

**Truth Table for ¬**

| P | ¬P |
|---|---|
| TRUE | FALSE |
| FALSE | TRUE |

**Truth Table for ∧**

| P | Q | P ∧ Q |
|---|---|---|
| TRUE | TRUE | TRUE |
| TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE |
| FALSE | FALSE | FALSE |

**Truth Table for ∨**

| P | Q | P ∨ Q |
|---|---|---|
| TRUE | TRUE | TRUE |
| TRUE | FALSE | TRUE |
| FALSE | TRUE | TRUE |
| FALSE | FALSE | FALSE |

- ¬ flips the value
- ∧ takes the 'worst' value
- ∨ takes the 'best' value





# The Basics
## Step 1: The Reference Columns

- We are going to construct a truth table for

(1) Cube(a) ∨ ¬Cube(a)

First, some columns:

1. Reference Columns: columns for each atomic sub-sentence of (1)
2. A column for (1) itself

**Truth Table for (1)**

| Cube(a) | Cube(a) ∨ ¬Cube(a) |
|---|---|
| T | |
| F | |

Second, fill the reference columns w/truth values.

- One row for each unique logical possibility





# The Basics
## Step 2: Inside Out

**Truth Table for (1)**

| Cube(a) | Cube(a) ∨ ¬Cube(a) |
|---|---|
| T | F |
| F | T |

- Third, fill column beneath innermost connective ¬:
  - In the first row, Cube(a) is T so ¬Cube(a) is F
  - In the second row, Cube(a) is F so ¬Cube(a) is T

## The Basics
### Step 3: The Main Connective

**Truth Table for (1)**

| Cube(a) | Cube(a) ∨ ¬Cube(a) |
|---|---|
| T | T F |
| F | T T |

- Last, fill columns beneath outermost connective ∨:
  - In the first row, Cube(a) is T and ¬Cube(a) is F, so their disjunction is T
  - In the first row, Cube(a) is F and ¬Cube(a) is T, so their disjunction is T
- This column lists every logically possible truth value for (1)





## The Basics
### Summary: In General

**How to Construct a Truth Table for Any Sentence P**

1. **Reference Columns**: Draw a column for each atomic sub-sentence of P, these columns are called the reference columns and are filled with every possible combination of truth-values for the sub-sentences
2. **Inside Out**: Draw a column for P itself. Then fill in the column below P's innermost connective. Repeat for the next innermost connective, until you get to the main connective.
3. **Main Connective**: Fill in the column under the main connective. This row lists the possible truth values of P





## Reference Columns
### There is More to It...

- When you have more than one atomic sub-sentence, filling in the reference columns requires more thought
- Remember that each row of the reference columns lists a unique logical possibility
- Also remember that there is supposed to be a row for every unique possibility
- Okay, well how many rows would we need for a formula with 2 atomic sub-sentences?





## Reference Columns
### How Many Rows?

Let's figure it out:

- We have 2 atomic sub-sentences
- Each can have 2 different truth values (T,F)
- So there $2^2 = 4$ possible combinations of truth values and atomic sub-sentences
- Therefore, a table for a formula with 2 atomic sub-sentences needs 4 rows

**You Always Need $2^n$ Rows**

In general, if there are $n$ atomic sub-sentences of P then there will be $2^n$ possible assignments of truth values to those atomic sub-sentences, in which case the truth table for P should have $2^n$ rows.

# Another Example
## Step 1

- Let's construct a truth table for:
  (2) ¬(Cube(a) ∧ Cube(b))
- We need 2 reference columns:
  - In them, we need a row for each of the 4 logical possibilities
  - Cube(a) can be T and Cube(b) T; Cube(a) T and Cube(b) F, and so on.

**Truth Table for (2)**

| Cube(a) | Cube(b) | ¬(Cube(a) ∧ ¬Cube(b)) |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |





# Another Example
## Steps 2 & 3

**Truth Table for (2)**

| Cube(a) | Cube(b) | ¬(Cube(a) | ∧ | ¬Cube(b)) |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | F | T | T |
| F | T | T | F | F |
| F | F | T | F | T |

- Next, the innermost connective ¬:
  - It will flip each value of Cube(b)
- Now, the next innermost ∧:
  - ∧ takes worst value of the pair
- Finally, the main connective ¬:
  - ¬ flips the value of the conjunction we just computed





# Building Reference Columns
## A Helpful Routine

- You need to list each possibility exactly once when filling in reference columns
- Here's a helpful routine for this:
  1. In the innermost ref. column, you alternate T's and F's
  2. In the next innermost column, you **double** that alternation, and so on for any more rows

**Truth Table for (2)**

| Cube(a) | Cube(b) | ¬(Cube(a) ∧ ¬Cube(a)) |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

Let's put this routine to work





# Yet Another Example
## Step 1

We'll construct a table for:

(3) (Cube(a) ∧ ¬Cube(b)) ∨ ¬Cube(c)

Let A = Cube(a), B = Cube(b), C = Cube(c)

**Table for (3)**

| A | B | C | (A ∧ ¬B) ∨ ¬C |
|---|---|---|---|
| T | T | T | |
| T | T | F | |
| T | F | T | |
| T | F | F | |
| F | T | T | |
| F | T | F | |
| F | F | T | |
| F | F | F | |

- First, the 8 rows of the reference columns:
  1. Alternate on innermost column
  2. Double this alternation on the next
  3. Double again

## Yet Another Example
Steps 2 & 3

**Table for (3)**

| A | B | C | (A | ∧ | ¬B) | ∨ | ¬C |
|---|---|---|---|---|---|---|---|
| T | T | T | | F | F | | F | F |
| T | T | F | | F | F | | T | T |
| T | F | T | | T | T | | T | F |
| T | F | F | | T | T | | T | T |
| F | T | T | | F | F | | F | F |
| F | T | F | | F | F | | T | T |
| F | F | T | | F | T | | F | F |
| F | F | F | | F | T | | T | T |

- Next, rows for innermost connectives (¬)
  - ¬B: flips value of B
  - ¬C: flips value of C
- Now, the row for the next innermost connective (∧)
  - ∧ takes lowest value
- Finally, the row for the main connective (∨):
  - ∨ takes highest value





## You Try It
In Class Exercise

Break into groups of 4-6 and construct a truth table for:

(4) (B ∨ ¬C) ∧ ¬A





## Boole
An Introduction

- *LPL* contains a program called Boole, which is for constructing truth tables
- Now that you've done a one by hand, you can appreciate how nice of a tool this is!
- Let's run through the basics of Boole by using it to construct a table for (3)





## Truth Tables
Summary

- Okay, we've learned how to draw these pretty tables, but how do they help us do logic?
  1. Truth tables probe logical possibility
  2. This underlies several important concepts:
     - Logical truth
     - Logical consequence
     - Logical equivalence
- Let's see how

# Logical Truth
## The Basics

**Logical Truth**

P is a logical truth if and only if it is **logically necessary**. That is, it is **not possible** for the laws of logic to hold while P is false.

- Logical truths are those sentences which are guaranteed by logic alone to be true
- Logical possibility is different from physical & other kinds of possibility
  - Cube(a) ∨ ¬Cube(a) vs. Cube(a) ∨ Tet(a) ∨ Dodec(a)
  - Traveling the speed of light vs. being a round square
- This sounds vague, can we do any better?
  - Yes, if we use truth tables





# Tautologies

**Tautology**

P is a tautology if and only if the truth table for P has only T's in the column under P's main connective

**Truth Table for (1)**

| Cube(a) | Cube(a) ∨ ¬Cube(a) | |
|---|---|---|
| T | T | F |
| F | T | T |

- So, (1) is a tautology
- Intuitively, is (1) a logical truth?
  - Yes!
- So, it looks like the idea of a tautology is a way of making the idea of a logical truth a bit more precise
- This is because truth tables are a precise way of thinking about logical possibility





# Tautologies
## Some Examples

- Cube(c) ∨ ¬Cube(c) is a tautology
- ¬(Tet(a) ∧ ¬Tet(a)) is a tautology
- Tet(a) ∨ Cube(a) is not a tautology





# Tautologies vs. Logical Truths

- Recall that logical truths are sentences which are guaranteed to be true by the laws of logic alone
- All tautologies are logical truths
- But are all logical truths tautologies?
  - In other words, can we just replace the idea of a logical truth with that of a tautology?
- No! Seeing this actually takes a little creativity

## A Curious Logical Truth
Which isn't a Tautolgy

- Surely it is a logical truth that Jay is Jay and that Kay is Kay:

  (5) $\mathsf{j} = \mathsf{j} \wedge \mathsf{k} = \mathsf{k}$
- But consider the truth table for (5):

**Truth Table for (5)**

| $\mathsf{j} = \mathsf{j}$ | $\mathsf{k} = \mathsf{k}$ | $\mathsf{j} = \mathsf{j} \wedge \mathsf{k} = \mathsf{k}$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

- First, reference columns
- Second, main connective
- But wait, there are F's in that column!
- When you build reference columns, you just list all the combinations
- But, some combinations don't make sense!





## Tautologies & Logical Truths
Summary

**Remember**

1. $\mathsf{P}$ is a tautology if and only if every row of the truth table assigns T to $\mathsf{P}$
2. If $\mathsf{P}$ is a tautology, then $\mathsf{P}$ is a logical truth
3. But some logical truths are not tautologies
4. $\mathsf{P}$ is TT-possible if and only if at least one row of its truth table assigns T to $\mathsf{P}$

Let's do exercise **4.5**





## Summary
What We Just Did

- We learned how to construct truth tables, by hand & with Boole
- We then applied truth tables to the problem of precisely defining:
  1. Logical truth
- We came up with a similar concept:
  1. Tautology
- We saw that all Tautologies are logical truths, but some logical truths are not tautologies





## Equivalence
Two Varieties

**Logical Equivalence**

Two sentences are logically equivalent if and only if they have the same truth values in every possible situation

- For example: $\mathsf{Tet(a)}$ and $\neg\neg\mathsf{Tet(a)}$ are logically equivalent

**Tautological Equivalence**

Two sentences are tautologically equivalent just in case the columns under their main connectives in a joint truth table are identical

- What is a joint truth table?

# Equivalence
## Joint Truth Tables

- The idea of a joint truth table is quite simple, just add a column on the right for another formula and calculate as before

**Joint Truth Table**

| P | Q | ¬ | (P ∧ Q) | ¬P | ∨ | ¬Q |
|---|---|---|---|---|---|---|
| T | T | F | T | F | F | F |
| T | F | T | F | F | T | T |
| F | T | T | F | T | T | F |
| F | F | T | F | T | T | T |

1. Ref. columns
2. Inner connectives
3. Main connectives

- The columns under the main connectives are identical
- So, these two sentences are tautologically equivalent





# Equivalence
## Logical vs. Tautological

- We started by characterizing two kinds of equivalence
  1. Logical
  2. Tautological
- Every two sentences that are tautologically equivalent are logically equivalent
- Does the reverse hold?
- No, this pair is logically equivalent:

(6) $a = b \land \text{Cube}(a)$

(7) $a = b \land \text{Cube}(b)$

- But we can show with Boole that they aren't tautologically equivalent





# Consequence
## Two Varieties Again

**Logical Consequence**

C is a logical consequence of $P_1, \ldots, P_n$ just in case it is logically impossible for C to be false while $P_1, \ldots, P_n$ are true

- We've already met this concept of validity/consequence
- It doesn't help us much with figuring out whether an argument is valid
- Proof provides one method, truth tables another:

**Tautological Consequence**

C is a tautological consequence of $P_1, \ldots P_n$ just in case every row in their **joint truth table** that lists T under $P_1, \ldots, P_n$ also lists T under C





# Tautological Consequence
## An Example

**Argument 1**

A ∨ B

¬A

---

B

**Joint Truth Table for Argument 1**

| A | B | A ∨ B | ¬A | B |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | T | F | F |
| F | T | T | T | T |
| F | F | F | T | F |

- First two columns for the premises
- Last column for conclusion
- Every row where both premises are T, the conclusion is T
- So B is a tautological consequence of A ∨ B and ¬A

# Tautological Consequence
## Another Example: Exercise 4.20

Let's run through exercise 4.20





# Tautological Consequence
## In-Class Exercise

Exercise **4.21**

| Taller(claire, max) ∨ Taller(max, claire)
| Taller(claire, max)
|---
| ¬Taller(max, claire)





# Consequence
## Questions

1. If C is a tautological consequence of $P_1, \ldots, P_n$, is C a logical consequence of $P_1, \ldots P_n$?
   - Yes, clearly
2. If C is a logical consequence of $P_1, \ldots, P_n$, is C a tautological consequence of $P_1, \ldots P_n$?
   - No, lets show it using Boole to construct a joint truth table for this argument:

**Argument 2**

| $a = b \land b = c$
|---
| $a = c$





# Taut Con
## Tautological Consequence in Fitch

- Truth tables provide a powerful but purely mechanical procedure to test for logical consequence
- But, they often get really tedious and long
- But, that's what computers are good at
- Fitch has a built-in mechanism for testing for tautological consequence
  - **Taut Con**
- Much like **Ana Con**, this is not a rule of inference, but a computational mechanism
- Let's run through exercise 4.26

# Summary
## What we did today

- We learned how to construct truth tables, by hand & with Boole
- We applied truth tables to the problem of precisely defining:
  1. Logical truth
  2. Logical equivalence
  3. Logical consequence
- We came up with three similar concepts:
  1. Tautology
  2. Tautological equivalence
  3. Tautological consequence
- In each case Tautological implied Logical, but Logical did not imply Tautological