

The Boolean Connectives

Negation, Conjunction & Disjunction

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Outline

- 1 Meet the Booleans
- 2 Ambiguity & Parentheses
- 3 Some Equivalences
- 4 Translation
- 5 Summary

Announcements

For 02.03

- 1 HW2 is due **now**
 - You should have some written exercises to hand-in

Today's Topic

Overview

- We've only looked at the logic of simple sentences like:
 - Mars is red*
 - Red(mars)
- But more complicated sentences have equally interesting logical properties
 - Sentences in which simple ones are modified:
 - *It is not the case that Mars is red*
 - Etc.
 - Sentences which compound **multiple** simple ones:
 - *Bill is rad and Logan is bogus*
 - *Jay is home or Kay is home*
 - Etc.
- These all involve a **connective** of some sort

Today's Topic

- Today we'll begin to learn about the logic of these connectives:

The Boolean Connectives

- ① *not, it is not the case that* (Negation)
 - ② *and* (Conjunction)
 - ③ *or* (Disjunction)
- Today we'll:
 - Learn a bit about what these words mean
 - Learn about some new symbols in FOL that are used to represent them
 - Discuss some issues these symbols raise
 - Discuss the connection between English sentences & FOL sentences with these symbols

Connectives & Meaning

Truth-Functions & Logic Games

- There are two ways we will explore the meaning of the Boolean connectives:
 - ① Truth-Functions
 - ② Logic Games
- Before getting into the details of today's topics, let's go over a few helpful general facts about these approaches

Truth-Functions

Parts & Wholes

- When considering the meaning of a declarative sentence one important thing to consider is its **truth conditions**
 - That is, the most general conditions under which it would be true

Truth-functions

The idea behind **truth-functions** is that the truth-value of a **whole sentence** can be computed solely from the truth value of it's **simplest parts**

Truth-Functions

The Boolean Connectives as Truth-Functions

- This idea has been applied to analyzing the meaning of the Boolean connectives
- For example, in the case of *and*:
 - (1) *Kay ran and Jay ran*
The truth value of (1) can be computed from the truth value of it's two **simplest** sentences
 - (1a) *Kay ran*
 - (1b) *Jay ran*
- The same approach can be pursued for Negation and Disjunction
- We will see these approaches in detail momentarily

Truth-Functions

Historical Notes

- The truth-functional approach to meaning originated with George Boole and Gottlob Frege, with important contributions along the way by Alfred Tarski
- It survives in modern philosophical theorizing about language and the mind:
 - Donald Davidson
 - Richard Montague
 - Jerry Fodor
 - And many others
- It also plays an important role among contemporary linguists who study meaning & communication

Logic Games

Meanings, Strategies & Games

- Another way of thinking about the meaning of a complex sentence draws on the idea of a **game**
- Imagine Jay and Kay disagree about the truth value of a complex sentence
 - They can resolve their disagreement by repeatedly challenging each other to justify their claims in terms of simpler claims, until finally their disagreement is reduced to a simple atomic claim
 - At that point, they can just examine the world to see who is right (ideally)
- These successive challenges can be thought of as a game where one player will win & one will lose
- On this approach, the meaning of a connective can be identified with the winning strategy for a game based on a sentence containing that connective

Logic Games

Historical Notes

- The game-theoretic approach to the meaning of the connectives originated with Jaakko Hintikka and Leon Henkin
 - It drew on philosophical ideas about mathematics, mind & language from:
 - Ludwig Wittgenstein (late work)
 - L.E.J. Brouwer
 - And it survives in more modern philosophical discussions of mathematics, mind & language:
 - Michael Dummett
 - Robert Brandom
 - Jaakko Hintikka
 - John Searle
 - And many others

Negation

English & FOL

- In English, negation is expressed in a number of ways:
 - 1 *not*
 - 2 *it is not the case that...*
 - 3 *non-*
 - 4 *un-*
- In FOL negation is expressed with one symbol: \neg
- \neg attaches to the front of any formula to produce a new one:
 - $\neg \text{Bizarre}(\text{jay})$
 - $\neg \text{SameSize}(a, b)$
 - $\neg \neg \text{SameSize}(a, b)$

Negation

English vs. FOL

- The grammar for \neg is sometimes the same as negation in English:
 - *It is not the case that Jay is bizarre* $\rightsquigarrow \neg\text{Bizarre}(\text{jay})$
- But it can be very different:
 - *Jay is not bizarre* $\rightsquigarrow \neg\text{Bizarre}(\text{jay})$
 - *Kay is unpopular* $\rightsquigarrow \neg\text{Popular}(\text{kay})$
- Okay, but what does negation **mean**?

Conjunction

English & FOL

- In English, conjunction is expressed in a number of ways:
 - 1 *and*
 - 2 *moreover*
 - 3 *but*
- In FOL, conjunction is expressed with one symbol: \wedge
- \wedge connects two sentences of FOL to form a new one:
 - $\text{Large}(a) \wedge \text{Cube}(a)$
 - $\text{Large}(a) \wedge \neg\text{Cube}(a)$
 - $\text{Large}(a) \wedge \neg\text{Cube}(a) \wedge \text{Small}(b)$

The Meaning of Negation

Truth-Functions & Games

Truth-Table for \neg

P	$\neg P$
TRUE	FALSE
FALSE	TRUE

- When **P** is true, $\neg P$ is false
- When **P** is false, $\neg P$ is true
- The truth value of $\neg P$ is a function of P's truth value

Game Rule for \neg

- 1 If you commit to the truth of $\neg P$, you are committed to the falsity of P
- 2 If you commit to the falsity of $\neg P$, you are committed to the truth of P

Let's absorb this a bit more w/the **You try it** from §3.1

Conjunction

English vs. FOL

- The grammar for \wedge is sometimes the same as conjunction in English:
 - *D.M.C is loud and Jam-Master Jay is proud*
 $\rightsquigarrow \text{Loud}(\text{dmc}) \wedge \text{Proud}(\text{jmj})$
- But it is usually **very** different:
 - *Brittany is deranged and delirious*
 $\rightsquigarrow \text{Deranged}(\text{brittany}) \wedge \text{Delirious}(\text{brittany})$
 - *Bill and Ted had an excellent adventure*
 $\rightsquigarrow \text{ExAdventure}(\text{bill}) \wedge \text{ExAdventure}(\text{ted})$

The Meaning of Conjunction

Truth-Functions & Games

Truth-Table for \wedge

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

- $P \wedge Q$ is true when **P is true and Q is true**
- Otherwise, $P \wedge Q$ is false
- The truth value of $P \wedge Q$ is a function of the truth values of P and Q

Game Rule for \wedge

- 1 If you commit to the truth of $P \wedge Q$, you commit to the truth of both P and Q
- 2 If you commit to the falsity of $P \wedge Q$, you commit to the falsity of either P or Q

Let's solidify this w/Exercise 3.5

Disjunction

English & FOL

- In English, disjunction is expressed with *or*
- In FOL, disjunction is expressed with \vee
- \vee connects two sentence of FOL to form a new one:
 - $\text{Cube}(a) \vee \text{Tet}(a)$
 - $\text{Cube}(a) \vee \neg \text{Tet}(a)$
 - $\text{Cube}(a) \vee \neg \text{Tet}(a) \wedge \text{Small}(a) \vee \neg \text{Large}(a)$

Disjunction

English vs. FOL

- The grammar for \vee is sometimes the same as disjunction in English:
 - *Mexico is beautiful or I drank too much tequila*
 $\rightsquigarrow \text{Beautiful}(\text{mexico}) \vee \text{2muchTequila}(\text{ws})$
- But it is often **very** different:
 - *Bill will pass or fail*
 $\rightsquigarrow \text{Pass}(\text{bill}) \vee \text{Fail}(\text{bill})$
 - *Bill or Ted will party*
 $\rightsquigarrow \text{Party}(\text{bill}) \vee \text{Party}(\text{ted})$

The Meaning of Disjunction

Truth-Functions & Games

Truth-Table for \vee

P	Q	$P \vee Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- $P \vee Q$ is false when both **P and Q are false**
- Otherwise, $P \vee Q$ is true
- The truth value of $P \vee Q$ is a function of the truth values of P and Q

Game Rule for \vee

- 1 If you commit to the truth of $P \vee Q$, you must commit to the truth of either P or Q
- 2 If you commit to the falsity of $P \vee Q$, you commit to the falsity of both P and Q

Let's solidify this w/Exercise 3.8

Ambiguity

What?

(2) *Max is home or Claire is home and Carl is happy*

- (2) is ambiguous (it has multiple interpretations):
 - Max or Claire is home and Carl is happy*
 - Either Max is home, or Claire is home and Carl is happy*
- You might try to translate (2) into FOL as:

(3) $\text{Home}(\text{max}) \vee \text{Home}(\text{claire}) \wedge \text{Happy}(\text{carl})$
- (3) is just as ambiguous as (2)!
- FOL aims to eliminate this ambiguity

Ambiguity

Introducing Parentheses

(3) * $\text{Home}(\text{max}) \vee \text{Home}(\text{claire}) \wedge \text{Happy}(\text{carl})$

- FOL outlaws ambiguity by considering sentences like (3) **ungrammatical**
 - (3) is like a sentence fragment in English
- To make (3) grammatical **parentheses** can be added in two ways:

(3a) $(\text{Home}(\text{max}) \vee \text{Home}(\text{claire})) \wedge \text{Happy}(\text{carl})$

(2a) *Max or Claire is home and Carl is happy*

(3b) $\text{Home}(\text{max}) \vee (\text{Home}(\text{claire}) \wedge \text{Happy}(\text{carl}))$

(2b) *Either Max is home, or Claire is home and Carl is happy*

Parentheses Matter

Non-Equivalence

- Parentheses matter to meaning
 - These two sentences are **not** equivalent:

(4) $\text{Tet}(\text{a}) \vee (\text{Tet}(\text{b}) \wedge \text{Tet}(\text{c}))$

(5) $(\text{Tet}(\text{a}) \vee \text{Tet}(\text{b})) \wedge \text{Tet}(\text{c})$
 - These two aren't equivalent either:

(6) $\neg(\text{Large}(\text{a}) \vee \text{Large}(\text{b}))$

(7) $\neg\text{Large}(\text{a}) \vee \text{Large}(\text{b})$
- Let's see this in more detail by doing exercise **3.12**

Parentheses

The Analogy With Arithmetic: Grouping Can Matter

- Different groupings w/parentheses in FOL can yield different truth conditions
 - This is just like **orders of operation** in arithmetic:

• $2 + (3 \times 4) = 14$	• $\text{Tet}(\text{a}) \wedge (\text{Tet}(\text{b}) \vee \text{Tet}(\text{c}))$
• $(2 + 3) \times 4 = 20$	• $(\text{Tet}(\text{a}) \wedge \text{Tet}(\text{b})) \vee \text{Tet}(\text{c})$
• Different!	• Different!
• $-(2 + 3) = -5$	• $\neg(\text{Tet}(\text{a}) \wedge \text{Tet}(\text{b}))$
• $-2 + 3 = 1$	• $\neg\text{Tet}(\text{a}) \wedge \text{Tet}(\text{b})$
• Different!	• Different!

Parentheses

The Analogy With Arithmetic: Sometimes Grouping Doesn't Matter

- But, just like arithmetic sometimes grouping does **not** matter:
 - + is associative:
 - $2 + (3 + 4) = 9$
 - $(2 + 3) + 4 = 9$
 - Same!
 - \wedge is too:
 - $\text{Tet}(a) \wedge (\text{Tet}(b) \wedge \text{Tet}(c))$
 - $(\text{Tet}(a) \wedge \text{Tet}(b)) \wedge \text{Tet}(c)$
 - Same!
 - \times is associative:
 - $2 \times (3 \times 4) = 24$
 - $(2 \times 3) \times 4 = 24$
 - Same!
 - \vee is too:
 - $\text{Tet}(a) \vee (\text{Tet}(b) \vee \text{Tet}(c))$
 - $(\text{Tet}(a) \vee \text{Tet}(b)) \vee \text{Tet}(c)$
 - Same!

Parentheses

Official Policy

Our Policy on Parentheses

Parentheses must be used whenever ambiguity would result from their omission. In practice, this means that conjunctions & disjunctions must be 'wrapped' in parentheses whenever combined by means of some other connective

- This allows us to omit parentheses when unnecessary, but requires us to include them when they are!
- This will make our formulas look as simple as possible

Equivalences

DeMorgan's Laws & Double Negation

- Just like any language, there are many ways to say the same thing in FOL
- Here are three you should be aware of

Important FOL Equivalences

- 1 **Double Negation:** $\neg\neg P \Leftrightarrow P$
- 2 **DeMorgan:** $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
- 3 **DeMorgan:** $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
- 4 There are many others we'll discover along the way

Let's examine DeMorgan's Equivalences by doing exercise 3.16

Translation

An Outline

- Translating from English to FOL is a useful ability
 - But, it can take some practice & skill
- Today we'll learn:
 - 1 When we will consider a translation to be correct
 - 2 Some tricks for translating conjunctions, disjunctions & negations

Translation

When is One Correct?

Correctness Condition for Translations

In order for a FOL sentence to be a good translation of an English sentence, it is sufficient that the two sentences have the same truth values in all possible circumstances, that is, that they have the same **truth conditions**

- Note that it is **not** sufficient for the two sentences have the same truth value in some **particular** world or situation

Translation

Varieties of Conjunction

- There are fine-grained differences between *and*, *but*, *however*, *yet* & *nonetheless*
- But because we are only interested in truth conditions they will all be translated as \wedge
- So, for example (8)-(12):
 - (8) *Jay is large and Kay is in charge*
 - (9) *Jay is large but Kay is in charge*
 - (10) *Jay is large however Kay is in charge*
 - (11) *Jay is large yet Kay is in charge*
 - (12) *Jay is large, Kay is in charge*
 Are all translated as:
 - (13) **Large(jay) \wedge InCharge(kay)**

Translation

Both

- Both* is often used to clarify where exactly the conjunction is:
 - A good translation of:
 - (14) *It is not true that Claire and Max are both at home*
 Is:
 - (15) $\neg(\text{Home}(\text{claire}) \wedge \text{Home}(\text{max}))$
 As opposed to:
 - (16) $\neg\text{Home}(\text{claire}) \wedge \text{Home}(\text{max})$
- A similar device exists for disjunction

Translation

Either Or

- Either...or* acts like our parentheses in FOL:
- Consider:
 - (17) *Either **a** is small and a cube or it is large*
 A good translation of this is:
 - (18) $(\text{Small}(\text{a}) \wedge \text{Cube}(\text{a})) \vee \text{Large}(\text{a})$
 As opposed to:
 - (19) $\text{Small}(\text{a}) \wedge (\text{Cube}(\text{a}) \vee \text{Large}(\text{a}))$

Translation

Neither Nor

- Another common construction involving disjunction is *neither...nor*
- These sentences can be translated as *not...or*, with *neither* marking the beginning of the disjunction
- Consider:

(20) *Neither Jay nor Kay is home*

A good translation of this is:

(21) $\neg(\text{Home}(\text{jay}) \vee \text{Home}(\text{kay}))$

As opposed to:

(22) $\neg\text{Home}(\text{jay}) \vee \text{Home}(\text{kay})$

Translation

Summary

- 1 We will consider a translation is **correct** when it correctly captures the target sentence's **truth conditions**
- 2 There are a variety of negations, all translated using \neg
- 3 There are a variety of conjunctions, all of which translate using \wedge
 - *and, but, however, yet & nonetheless*
- 4 To translate *or*, use \vee
- 5 *Either* and *both* are used to indicate grouping

Let's solidify these points by doing parts of exercises **3.21** & **3.22**

Summary

Today's Class

- We met the Booleans:
 - 1 **Conjunction:** *and*, \wedge
 - 2 **Disjunction:** *or*, \vee
 - 3 **Negation:** *not*, \neg
- We learned how parentheses in FOL make sure that \neg , \vee , \wedge play nicely together
- We saw a few important equivalences
 - **Double Negation:** $\neg\neg P \Leftrightarrow P$
 - **DeMorgan:** $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
 - **DeMorgan:** $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Summary

Today's Class (Cont'd)

- We picked up a few tricks for translating from English to FOL:
 - Correct translations capture truth conditions
 - Various important facts about translating conjunctions, disjunctions & negations:
 - There are many forms of conjunction in English, but all get translated as \wedge
 - *Both* is used to clarify the location of a conjunction
 - *Either...or* does the same for disjunction
 - *Neither...nor* is translated as *not...or*

Homework 3

Due 02.10

Homework 3

Due by **Tuesday 02.10**:

- Exercises 3.6, 3.9, 3.23, 4.2, 4.7
- Remember to read §4.1 of the textbook for next class