

## Announcements

10.20

## Introduction to Quantification

 $\forall$  and  $\exists$ , in more detail

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- ① Wednesday section is on indefinite hiatus
- ② I will hold an additional office hour instead
- ③ Monday from 5-6pm, Goldwin Smith 237

## Outline

- ① Introduction & Review
- ② Semantics

## Quantities

In Thought &amp; Talk

- In our daily lives, we think & talk about **quantities**
  - All money
  - No ex-girlfriend
  - Two (many) siblings
  - One friend
  - Few enemies
- This thought & talk is governed by interesting **logical principles**
- These principles cannot be captured with the truth-functional connectives
- But we still want to capture them!

# Quantifiers

## And Quantifier Phrases

- (1) **Some words** are wasted
  - (2) **Every accountant** is a vampire
  - (3) **Three cats** are sleeping
  - (4) **Several parents** showed up to George's party
  - (5) **No friends** came to my party
- The above sentences contain **quantifier phrases**
  - Simple **quantifier phrases** have two parts:
    - ① A **quantifier**
    - ② A **noun**
  - Last class, we learned how to represent quantifiers and quantifier phrases in FOL

# Quantifiers in FOL

## Meet $\forall$ and $\exists$

- We added the **quantifier symbols** for FOL:
  - The Universal Quantifier**  $\forall$  (*everything*)
  - The Existential Quantifier**  $\exists$  (*something*)
- And **variables**
  - FOL has infinitely many variables:  
 $t, u, v, w, x, y, z, t_1, \dots, t_n, u_1, \dots, u_n, v_1, \dots, v_n, \dots$
  - They go in the slots of predicates:  
 $\text{Cube}(y), \text{FrontOf}(u, v), \text{Between}(z, u_{21}, w)$
- Together, these two resources allowed us to represent quantificational sentences

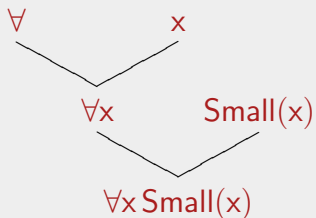
# The Universal Quantifier

## Universal Statements

- How do you represent a **universal statement** in FOL?

### Example

*Everything is small:*



- ① It's a universal statement, so use  $\forall$
- ② Pick a variable to use, like  $x$
- ③ Pair  $\forall$  with that variable
- ④ Plug that variable into the predicate of the claim
- ⑤ Stick together the two things you've made

- We read  $\forall x \text{Small}(x)$  as *For every object  $x$ ,  $x$  is small*
- This is an intuitively correct paraphrase of *Everything is small*

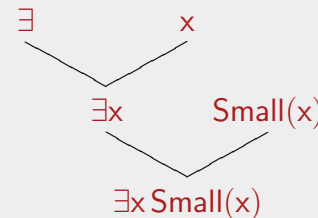
# The Quantifier

## Existential Statements

- How do you represent a **Existential statement** in FOL?

### Example

*Something is small:*



- ① Its a existential statement, so use  $\exists$
- ② Pick a variable to use, like  $x$
- ③ Pair  $\exists$  with that variable
- ④ Plug that variable into the predicate of the claim
- ⑤ Stick together the two things you've made

- We read  $\exists x \text{Small}(x)$  as *For some object  $x$ ,  $x$  is small*
- This is an intuitively correct paraphrase of *Something is small*

# Variables

Complete vs. Incomplete

- There's a big difference between these two formulas:
  - (6)  $\text{Small}(x)$
  - (7)  $\text{Small}(a)$
- (7) makes a claim that is **true or false**
  - Either  $a$  is small or it isn't
- (6) does **not**
- (6) is an **incomplete** claim
  - It's like saying *it is small* without telling us what *it* is!
- However, (6) becomes complete when  $\exists x$  or  $\forall x$  is added
- ' $\exists x \text{Small}(x)$ ' is either true or false

# Variables

Complete vs. Incomplete?

## A Question

When exactly does a formula containing variables make a complete claim?

- Does (8) make a complete claim?
  - (8)  $\exists x (\text{Small}(x) \wedge \text{Cube}(x))$
- What about (9)?
  - (9)  $\exists x (\text{Small}(x) \wedge \text{Cube}(x)) \vee \text{LeftOf}(x, a)$

## The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the **scope** of a quantifier symbol attached to that variable

# Scope

Some Terminology

## Scope

- 1 A quantificational wff  $\forall v A$  is formed by sticking together some wff  $A$  and quantifier-phrase  $\forall v$
- 2 We call  $A$  that quantifier's **scope**.
  - $\forall x (\text{Small}(x) \wedge \text{Tet}(x))$ 
    - $\forall x$ 's Scope:  $\text{Small}(x) \wedge \text{Tet}(x)$
  - $\forall x \text{Small}(x) \wedge \text{Tet}(x)$ 
    - $\forall x$ 's Scope:  $\text{Small}(x)$

# Variables

Complete vs. Incomplete: Revisited

## The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the **scope** of a quantifier symbol attached to that variable

- Does (8) make a complete claim?
  - (8)  $\exists x (\text{Small}(x) \wedge \text{Cube}(x))$ 
    - Yes; both occurrences within scope of  $\exists x$
- What about (9)?
  - (9)  $\exists x (\text{Small}(x) \wedge \text{Cube}(x)) \vee \text{LeftOf}(x, a)$ 
    - No; 3rd occurrence outside scope of  $\exists x$

# Binding

## More Terminology

### Bondage

An occurrence of a variable  $v$  is **bound** iff  $v$  occurs within the scope of either  $\forall v$  or  $\exists v$

- 1st & 2nd occurrences of  $x$  are bound; 3rd is not  
(9)  $\exists x (\text{Small}(x) \wedge \text{Cube}(x)) \vee \text{LeftOf}(x, a)$

### Freedom

An occurrence of a variable  $v$  is **free** iff  $v$  does **not** occur within the scope of either  $\forall v$  or  $\exists v$

- 3rd occurrence of  $x$  in (9) is free; 1st & 2nd are not

# Two More Things

## A New Version of The Answer & Wffs vs. Sentences

### The Answer (Second Version)

A formula containing variables makes a complete claim just in case every variable is bound

### Sentences vs. Wffs (Approximation)

- 1 *Well-formed formulas* or *wffs* is the set of **all** grammatical expressions of FOL, including both incomplete claims, like 'Tet( $x$ )' and complete ones
- 2 *Sentences* are formulas that make complete claims; contain no variables or only bound ones

# Wffs v. Non-Wffs

## Some Examples

### Wffs

- 1 Tet( $a$ )
- 2 Cube( $y$ )
- 3 (Cube( $y$ )  $\wedge$  Tet( $a$ ))
- 4 ( $\exists y$  (Cube( $y$ )  $\wedge$  Tet( $a$ )))
- 5 ( $\exists y$  Cube( $y$ ))  $\wedge$  Tet( $a$ )
- 6 Tet( $a$ )  $\rightarrow$  (Cube( $b$ )  $\wedge$  Small( $b$ ))

### Non-Wffs

- 1 Tet
- 2 ( $y$ )Cube
- 3 Cube( $y$ , Small)
- 4  $\wedge$ Cube( $y$ ) Tet( $a$ )
- 5  $\exists$ (Cube( $y$ )  $\wedge$  Large( $y$ ))
- 6 Tet( $a$ )  $\rightarrow$  Cube( $b$ )  $\wedge$  Small( $b$ )

- Now that we're clear on the wff v. non-wff distinction, let's draw the one we set out to draw
  - The wff v. sentence distinction

# Sentences v. Wffs

## Some Examples

### Non-Sentence Wffs

- 1 Tet( $y$ )
- 2  $\neg$ Cube( $y$ )
- 3 (Cube( $y$ )  $\wedge$  Tet( $a$ ))
- 4 (( $\exists y$  Cube( $y$ ))  $\wedge$  Tet( $y$ ))
- 5 ( $\exists y$  (Cube( $y$ )  $\wedge$  Tet( $x$ )))

- Free variables

### Sentences

- 1 Tet( $a$ )
- 2  $\neg$ Tet( $a$ )
- 3 (Cube( $a$ )  $\wedge$  Tet( $a$ ))
- 4 ( $\exists y$  (Cube( $y$ )  $\wedge$  Tet( $y$ )))
- 5 ( $\exists y$  (Cube( $y$ )  $\wedge$  ( $\exists x$  Tet( $x$ ))))

- No free variables

# Semantics & Quantification

Where we Are

- We know what the truth functional connectives ( $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ ) **mean**
  - Their **meanings** are given by their **truth tables**
  - **Terminology:** *semantics* is the study of **meaning**
- We have not yet learned the semantics for quantifier symbols ( $\forall, \exists$ )
- As it turns out, we **cannot** provide a semantics for quantifiers using **truth tables**
- Why?

# Semantics & Quantification

Why Not Truth Tables

- Truth tables work by explaining the truth of a complex formula in terms of the truth of its parts
  - Example:  $\neg P$  is T iff P is F
- The problem with using truth tables for quantifiers is that the truth of quantified formulas cannot be determined from the truth of its parts
  - Example:  $\forall x \text{Cube}(x)$  is T iff ???
    - $\text{Cube}(x)$  is T? F?
    - Neither!
    - $\text{Cube}(x)$  isn't capable of truth or falsity, it's too incomplete!
- So, we can't use truth tables to explain what quantified sentences mean

# Satisfaction

The Basic Idea

- If not truth tables, what?
- We'll use a method pioneered by Alfred Tarski (1936)
- He introduced the idea of an **object satisfying a formula**
- Here's the intuition behind satisfaction
  - Although a formula with a free variable like  $\text{Cube}(x)$  is neither true nor false, we can think of it being **true of some object  $o$**
  - Tarski called this special idea of being true of an object *satisfaction*
  - For example,  $o$  satisfies  $\text{Small}(x) \wedge \text{Cube}(x)$  iff  $o$  is a small cube

# Satisfaction

The Precise Definition

## Definition of Satisfaction

An object  $o$  satisfies a wff  $S(x)$  containing  $x$  as its only **free variable** iff the following two conditions are met:

- 1 If we give a  $o$  a name that's not taken, call it  $n_i$ , then  $S(n_i)$  is true
- 2  $S(n_i)$  is the result of replacing **every** occurrence of  $x$  in  $S(x)$  with  $n_i$

- Let's work through some examples in Tarski's World

# Domain of Discourse

The Things We're Talking About

- When we ask:
  - Is there an object  $o$  that satisfies  $S(x)$ ?
- Which objects should we look at?
- When we communicate, we take as given a collection of objects we're interested in talking about
- Sometimes this collection is absolutely all objects, but more commonly it is some restricted set of objects
- We'll call this set the *domain of discourse*
- So, the answer to our question above is: *the objects in the domain of discourse!*

# Domain of Discourse

An Example

## Example

- When I say *Every student is sleepy* here and now, which students does it seem most reasonable for me to be talking about?
- You! The students in this classroom (Sadly)
- The domain of discourse is taken to be set of things in this room
- When I say *every student* I restrict your attention to the students **in this room**
- In Tarski's World the domain of discourse is the collection of blocks on the board

# Existential Statements

When are They True?

- Now that we understand satisfaction, we can say when quantified statements are true
- Before an exact formulation, let's think through it
- *Something is blue* is true just in case there is some object  $o$  and  $o$  is blue
- Truth of  $\exists x \text{Blue}(x)$  can be determined similarly:
  - $\exists x \text{Blue}(x)$  is true iff **some** object  $o$  **satisfies**  $\text{Blue}(x)$
  - That is, if there is some object  $o$  such that when you give it an unused name  $n$ ,  $\text{Blue}(n)$  comes out true
  - If there is no such object,  $\exists x \text{Blue}(x)$  is false

# Existential Statements

Official Semantics

## Semantics for $\exists$

$\exists x S(x)$  is TRUE if and only if there is at least one object that **satisfies**  $S(x)$

## Example

When is  $\exists x (\text{Cube}(x) \wedge \text{Small}(x))$  true?

- By the semantics for  $\exists$ :
  - (10) There is at least one object that **satisfies**  $\text{Cube}(x) \wedge \text{Small}(x)$
- By the definition of **satisfaction** (10) amounts to:
  - When we give  $o$  some unused name  $n$ ,  $\text{Cube}(n) \wedge \text{Small}(n)$  comes out true

# Existential Statements

## Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some existential claims

# Universal Statements

## When are They True?

- When are universal statements true?
- Before the exact answer, let's get some intuitions
- *Everything is beautiful* is true just in case for every object  $o$ ,  $o$  is beautiful
- The truth of  $\forall x \text{ Beautiful}(x)$  is similar:
  - Consider whether every object  $o$  in the domain of discourse **satisfies**  $\text{Beautiful}(x)$
  - That is, for every object  $o$  see whether when you give it an unused name  $n$ ,  $\text{Beautiful}(n)$  comes out true
  - If so, then  $\forall x \text{ Beautiful}(x)$  is true
  - Otherwise, it is false
- Okay, let's see the precise semantics

# Universal Statements

## Official Semantics

### Semantics for $\forall$

$\forall x S(x)$  is TRUE if and only if **every** object **satisfies**  $S(x)$

### Example

When is  $\forall x (\text{Cube}(x) \wedge \text{Small}(x))$  true?

- By the semantics for  $\forall$ :
  - (11) Every object  $o$  **satisfies**  $\text{Cube}(x) \wedge \text{Small}(x)$
- By the definition of **satisfaction** (11) amounts to:
  - When we give each  $o$  some unused name  $n$ ,  $\text{Cube}(n) \wedge \text{Small}(n)$  comes out true

# Universal Statements

## Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some universal claims

# Summary

## Today's Highlights

### Summary

- 1 A variable is **bound** by a quantifier when it occurs in the quantifier's scope
- 2 A variable is **free** if it does not occur within the scope of a quantifier
- 3 A quantifier's **scope** is the wff it attaches to
- 4 Wffs with free variables are neither true nor false
- 5 Truth-tables don't work for quantifiers, they operate on something that is neither T nor F
  - A wff with a free variable
- 6 So the semantics of quantifiers requires a new idea:  
**satisfaction**