Notes on Thomason (1970)

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1 Thomason's Logic of Indeterminist Time (LIT)

Below I summarize the formal system presented in Thomason (1970)

1.1 Syntax

(1)

0.	$\alpha \in \mathcal{A}t = \{A_1, \ldots, A_n, \ldots\}$	$\Longrightarrow$	$\alpha \in \mathcal{W}\!f\!f$	*Atomic*
1.	$\phi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$[\mathrm{F}]\phi \in \mathcal{W}\!f\!f$	*Future Tense*
2.	$\phi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$[\mathrm{P}]\phi \in \mathcal{W}\!f\!f$	*Past Tense*
3.	$\phi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$[\mathrm{L}]\phi \in \mathcal{W}\!f\!f$	*Inevitability*
4.	$\phi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$[\mathrm{T}]\phi \in \mathcal{W}\!f\!f$	*Truth*
5.	$\phi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$\neg\phi \in \mathcal{W}\!f\!f$	*Negation*
6.	$\phi, \psi \in \mathcal{W}\!f\!f$	$\Longrightarrow$	$\phi \to \psi \in \mathcal{W}\!f\!f$	*Material Conditional*

1.2 Semantics

1.2.1 Models

(2) $\mathcal{M} = \langle \mathcal{T}, < \rangle$

1. $\mathcal{T}$ is a non-empty set of **times**
2. $<$ is a (binary) **temporal precedence** relation on $\mathcal{T}$. It satisfies three axioms:

Connectedness of the Past (Thomason 1970: 266)
$\forall t_1, t_2, t_3 \in \mathcal{T}$, if $t_2 \neq t_3$, $t_2 < t_1$ & $t_3 < t_1$ then $t_2 < t_3$ or $t_3 < t_2$

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Transitivity (Thomason 1970: 266)
$\forall t_1, t_2, t_3 \in \mathcal{T}$, if $t_1 < t_2$ & $t_2 < t_3$ then $t_1 < t_3$

Closure (Thomason 1970: 277)
$\forall t_1\ \exists t_2,\ t_1 < t_2$

1.2.2 Histories

Histories are maximal chains of times

(3) $\mathcal{H}$ is the set of **histories**, where h is a history iff 1-3 are met:

1. $h \subseteq \mathcal{T}$
   Histories are collections of times
2. $\forall t_1, t_2 \in h$, if $t_1 \neq t_2$ then $t_1 < t_2$ or $t_2 < t_1$
   Every time in a history is related by temporal precedence
3. If $h' \subseteq \mathcal{T}$ s.t.: $\forall t_1, t_2 \in h'$, if $t_1 \neq t_2$ then $t_1 < t_2$ or $t_2 < t_1$
   then, $h' = h$ if $h \subseteq h'$
   Histories are the biggest collections of related times

(4) If $t \in \mathcal{T}$ then $\mathcal{H}_t$ is the set of all **histories** h_t **containing** $\boldsymbol{t}$

1.2.3 Valuations, Truth & Supertruth

(5) $V(\alpha, t) : (\mathcal{A}t \times \mathcal{T}) \mapsto \{0, 1\}$ is an **atomic valuation**
Notation: $V_t(\phi) := V(\phi, t)$

(6) $V(\phi, t, h) : (\mathcal{W}\!f\!f \times \mathcal{T} \times \mathcal{H}) \mapsto \{0, 1\}$ is a (bivalent) **valuation** iff $t \in h$ & 1-6 hold:
Notation: $V_t^h(\phi) := V(\phi, t, h)$

1.	$V_t^h(\alpha)$	$=$	$V_t(\alpha)$
2.	$V_t^h(\phi \to \psi) = 1$	$\Longleftrightarrow$	$V_t^h(\phi) = 0$ or $V_t^h(\psi) = 1$
3.	$V_t^h(\neg\phi) = 1$	$\Longleftrightarrow$	$V_t^h(\phi)$
4.	$V_t^h([\mathrm{F}]\phi) = 1$	$\Longleftrightarrow$	$\exists t' \in h : V_{t'}^h(\phi) = 1$ & $t < t'$
5.	$V_t^h([\mathrm{P}]\phi) = 1$	$\Longleftrightarrow$	$\exists t' \in h : V_{t'}^h(\phi) = 1$ & $t' < t$
6.	$V_t^h([\mathrm{L}]\phi) = 1$	$\Longleftrightarrow$	$\forall h' \in \mathcal{H}_t : V_t^{h'}(\phi) = 1$
7.	$V_t^h([\mathrm{T}]\phi) = 1$	$\Longleftrightarrow$	$V_t^h(\phi) = 1$

We say ϕ is **true** at t relative to h iff $V_t^h(\phi) = 1$

(7) $\mathbb{V}(\phi, t) : F \times \mathcal{T} \mapsto \{1, 0\}$ is a **super-valuation** iff $F \subseteq \mathcal{W}\!f\!f$ & 1-2 hold:
Notation: $\mathbb{V}(\phi)_t := \mathbb{V}(\phi, t)$

1. $\mathbb{V}_t(\phi) = 1 \iff \forall h \in \mathcal{H}_t : V_t^h(\phi) = 1$
2. $\mathbb{V}_t(\phi) = 0 \iff \forall h \in \mathcal{H}_t : V_t^h(\phi) = 0$

Note: $\mathbb{V}_t(\phi)$ is otherwise undefined, so supervaluations aren't bivalent

We say ϕ is **supertrue** at t iff $\mathbb{V}_t(\phi) = 1$

1.2.4 Consequence & Validity

(8) $\Gamma \Vdash \phi \iff \forall \mathbb{V}_t, \mathcal{M}, t \in \mathcal{T}_{\mathcal{M}} : \mathbb{V}_t(\phi) = 1$ if $\forall \psi \in \Gamma : \mathbb{V}_t(\psi) = 1$

Consequence is preservation of supertruth in all models under all supervaluations

(9) $\Vdash \phi \iff \varnothing \Vdash \phi$

Validity is supertruth in all models under all supervaluations

2 Applications of LIT

- If we restricted ourselves to linear models, like $\mathcal{M}_1$, we would have $\models [\text{F}]\phi \rightarrow [\text{L}][\text{F}]\phi$ as well as $[\text{F}]\phi \models [\text{L}][\text{F}]\phi$

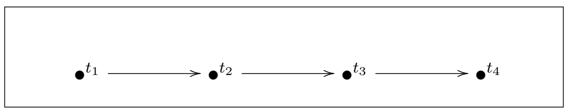

Fig. 1. $\mathcal{M}_1$: a linear model structure

- Both these facts seem to amount to determinism, which shouldn't follow as a matter of logic!
- To allow our tense logic to be indeterministic, LIT allows branching models like $\mathcal{M}_2$

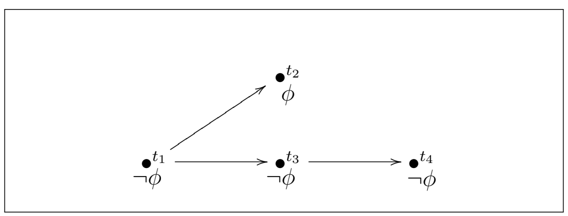

Fig. 2. $\mathcal{M}_2$: a branching model structure under some valuations

- $\mathcal{M}_2$ provides us with enough to show $\nVdash [\text{F}]\phi \rightarrow [\text{L}][\text{F}]\phi$
- In $\mathcal{M}_2$ there are two histories: $h_1 = \{t_1, t_2\}$ & $h_2 = \{t_1, t_3, t_4\}$

  - $V_{t_1}^{h_1}([\text{F}]\phi) = 1$, since $t_1 < t_2$ & $V_{t_2}^{h_1}(\phi) = 1$ (By 6.4)
  - $V_{t_1}^{h_1}([\text{L}][\text{F}]\phi) = 0$, because $V_{t_1}^{h_2}([\text{F}]\phi) = 0$ since $\nexists t \in h_2 : t_1 < t$ & $V_t^{h_2}(\phi) = 1$ (By 6.4,6.6)
  - So $\exists h : V_{t_1}^h([\text{F}]\phi \rightarrow [\text{L}][\text{F}]\phi) = 0$, namely h_1
    - ► Therefore, $\mathbb{V}_{t_1}(\phi)$ is undefined (By 7) & so $\nVdash [\text{F}]\phi \rightarrow [\text{L}][\text{F}]\phi$ (By 8)
- Interestingly: $[\text{F}]\phi \Vdash [\text{L}][\text{F}]\phi$
- Here's what Thomason (1970) says about it (using our notation):

> Here we have a case in which [material] implication differs from consequence; $[\text{L}][\text{F}]\phi$ is a consequence of $[\text{F}]\phi$ but does not imply $[\text{F}]\phi$. Intuitively this means that the argument from $[\text{F}]\phi$ to $[\text{L}][\text{F}]\phi$ is a valid one, for if it is already true that a thing will come to be, it is inevitable that it will come to be. But at the same time, it does not follow from supposing that a thing will come to be that it will inevitably come to be. To suppose that ϕ will be is to posit that we will be in a situation in which ϕ is true, that we will follow history h in which ϕ is sooner or later satisfied. But this is quite different from positing that such histories are the only alternatives now open; this would amount to positing that ϕ is inevitable. In our semantic theory this difference between supposing that ϕ will be and supposing that it is now true that ϕ will be is represented by the difference between making $[\text{F}]\phi$ and antecedent of a [material] implication as in 6.5 and making it a premise of the consequence relation as in 6.3.

- This sounds a lot like Aristotle when he maintains that *what is necessarily is, when it is* while maintaining that some future contingents are gappy

References

Thomason, R. H. (1970). 'Indeterminist Time and Truth-Value Gaps'. *Theoria*, **36**: 246–281.